INTRO TO SYMMETRIC FUNCTIONS

Here are summarized some definitions for the basic families of symmetric functions illustrated with Young tableaux.

1. Introduction

We are working with d variables $\{x_1, x_2, \ldots, x_d\}$.

1.1. **Partitions and notations.** The partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_L]$ is a tuple of positive integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L$, with $|\lambda| = \sum \lambda_i$, the *weight* and L the *length*. $|\lambda| = n$ is denoted by $\lambda \vdash n$.

An alternative notation is the *multiplicity notation*, $\rho = (1^{\rho_1} 2^{\rho_2} \cdots n^{\rho_n})$, with $\lambda \vdash n$. This indicates that in the tuple $\{\lambda_1, \ldots, \lambda_L\}$, the value i occurs ρ_i times. Note that

$$|\lambda| = \sum_j j\, \rho_j \qquad \text{and} \qquad L = \sum_j \rho_j \ .$$

Example. To illustrate, $\lambda = [55444322]$ has weight 29, length 8, with $\rho = (2^2 3^1 4^3 5^2)$.

A partition can be graphed as a *Ferrers diagram*: rows of dots with row i consisting of λ_i dots. And as well by a *tableau* consisting of boxes, which may be empty or filled with numbers, variables, etc. For example, $\lambda = [5311]$ is represented by

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & & & & \\ \bullet & & & & \end{array} \qquad \text{or} \qquad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & \\ \cline{1-3} \\ \cline{1-1} \\ \cline{1-1} \end{array}$$

A filled diagram may look something like

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 3 & 4 & 5 \\ \cline{1-3} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \tag{1}$$

In our context, we will be working with *semistandard Young tableaux*, abbreviated SSYT, meaning that the boxes are filled according to the following protocols:

1. Horizontal rows: the values are weakly increasing (nondecreasing) along the row, left to right
2. Vertical columns: the values are strictly increasing moving from top to bottom.

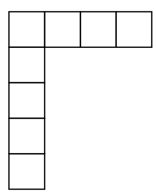
The tableau above, (1), is an SSYT.

1.2. **Monomial terms.** Given a set of variables $\{x_1, \ldots, x_d\}$, we correspond to an SSYT the monomial consisting of the product of x's with subscripts taken from the values in the boxes. Continuing with our example tableau, we have

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 3 & 4 & 5 \\ \cline{1-3} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \qquad \longleftrightarrow \qquad x_1^3 x_2 x_3^2 x_4^2 x_5^2$$

Using this correspondence, we can express the various basic types of symmetric functions using tableaux.

1.3. **Hooks.** A diagram for a partition of the form $\lambda = [n, 1, 1, 1, \ldots, 1] = (n1^j)$, $\lambda \vdash n + j$, is called a *hook*. For example, with $\lambda = [41111]$, we have



More generally, interior to any diagram are hooks, shaped similarly, i.e., consisting of all boxes to the right of, including, a given box and all boxes directly below it. The *hook lengths* count how many boxes comprise a given hook. Here is a hook diagram with hook lengths indicated in each box:

$$\begin{array}{|c|c|c|c|} \hline 8 & 3 & 2 & 1 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$$

and we have, e.g., for $\lambda = [5311]$,

$$\begin{array}{|c|c|c|c|c|} \hline 8 & 5 & 4 & 2 & 1 \\ \hline 5 & 2 & 1 \\ \cline{1-3} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \hline & \mathbf{x} & \\ \cline{1-3} \\ \cline{1-1} \\ \cline{1-1} \end{array}$$

with a typical hook indicated by x's.

The five families are discussed in the accompanying pages:

(1) Monomial
(2) Elementary
(3) Homogeneous
(4) Power sum
(5) Schur